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Generalized Berry phase for the most general time-dependent damped harmonic oscillator

Donald H Kobe

Department of Physics, University of North Texas, Denton, TX 76203-5368, USA

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Abstract. The most general time-dependent Hamiltonian for a harmonic oscillator is both linear and quadratic in the coordinate and the canonical momentum It describes in general a harmonic oscillator with a time-dependent mass, a time-dependent friction (or antifriction) 'constant', and a time-dependent spring 'constant', acted upon by a time-dependent force The energy, whose time derivative is the power, is in general different from the Hamiltonian. The generalized Berry phase for a given energy eigenstate has a state-dependent part, which vanishes if there is no damping, and an arbitrary state-independent part If the Hamiltonian is identified as the energy, both the energy eigenvalues and the generalized Berry phase are different In the adiabatic limit both approaches give the same total phase, which is the sum of the dynamical and Berry phases

1. Introduction

The Berry phase [1] for a time-dependent generalized harmonic oscillator Hamiltonian, which has a cross term (pq) between the generalized coordinate q and the canonical momentum p, has been obtained by a number of authors [2-6]. Jackiw [7] and Gerbert [8] showed that there is an ambiguity in the Berry phase because it is not invariant under unitary transformations. The Berry phase could even be removed by a suitable unitary transformation [9] In a previous paper [10], the Berry phase was generalized so that it is invariant under unitary transformations [11] This generalized Berry phase is also applicable to systems which are neither adiabatic nor cyclic. It therefore generalizes the phase of Aharonov and Anandan [12, 13], who define a gauge-invariant phase which is applicable to non-adiabatic processes. When applied to the generalized harmonic oscillator Hamiltonian, the generalized Berry phase is zero [10]. This behaviour can be understood since the generalized harmonic oscillator can be reduced by a gauge transformation to the standard Hamiltonian (with no cross term) for a simple harmonic oscillator with time-dependent mass and spring 'constant'. The generalized Berry phase for the standard Hamiltonian of a simple harmonic oscillator is known to be zero, and is invariant under gauge or unitary transformations.

In a recent paper, Engineer [14] further generalized the generalized harmonic oscillator Hamiltonian by adding a term proportional to the coordinate. He obtains a state-independent Berry phase, which he calls a 'parameter space analogue of the Aharonov-Bohm [15] effect'. For this Berry phase the physical space is topologically trivial, whereas the Aharonov-Bohm effect requires a non-trivial topology [16]. The state-independent Berry phase obtained by Engineer [14], using Berry's non-invariant formula [1], can be eliminated by a gauge transformation [17, 18] and is therefore unobservable.

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In this paper I consider the most general time-dependent harmonic oscillator Hamiltonian [19], which is both linear and quadratic in the coordinate and canonical momentum. In general the system described by this Hamiltonian is a harmonic oscillator with a time-dependent mass, a time-dependent spring 'constant', and a time-dependent friction (or antifriction) 'constant' that is acted upon by a time-dependent force This paper includes *both* a time-dependent mass and time-dependent damping (or antidamping) in a Hamiltonian formulation [20–24].

The interpretation of the Kanai Hamiltonian [21] as describing damping in quantum mechanics has been criticized [23] because of the apparent violation of the uncertainty principle. The uncertainty principle in terms of the canonical momentum is always satisfied, but in terms of the kinetic (or mechanical) momentum it is not in general satisfied. In this paper the friction 'constant' is generalized to be time-dependent and can be either positive, corresponding to damping, or negative, corresponding to antidamping. The environment can supply energy to a quantum particle as well as remove energy [24]. The phenomenological friction 'constant' can consequently be chosen so that the uncertainty principle in terms of the kinetic momentum is always satisfied.

The concept of a time-dependent mass in quantum mechanics may seem artificial, since the rest mass of particles is constant. Nevertheless, the effective mass of an electron in a solid can be time dependent if time-dependent external forces are applied to the solid so that the lattice constants change in time. Other examples of a time-dependent mass in quantum mechanics have been considered [25–27].

The energy of a time-dependent system is not necessarily the Hamiltonian [22, 28, 29]. The Hamiltonian determines the time development of the system, classically through Hamilton's equations and quantum mechanically through the Schrödinger equation. These equations are *form invariant* under gauge transformations, which implies that the Hamiltonian is gauge dependent. On the other hand, the energy, which is gauge invariant, has the property that its total time derivative is equal to the power transferred between the system and the environment. Because of damping, the kinetic (or mechanical) momentum is not equal to the canonical momentum. Likewise, the energy, which is the sum of the kinetic energy plus the conservative potential energy, is not equal to the Hamiltonian.

When a system is canonically quantized we can calculate the generalized Berry phase [10], which is invariant under unitary transformations and applicable to systems neither adiabatic nor cyclic. For the most general harmonic oscillator the generalized Berry phase has a state-dependent part, which vanishes if there is no damping (or antidamping), and an arbitrary state-independent part. The solution to the energy eigenvalue problem gives the eigenenergies, from which the dynamical phase is obtained. In the adiabatic limit the total phase is the dynamical phase plus the generalized Berry phase. The arbitrary state-independent part of the Berry phase is chosen such that the total phase has no state-independent part.

When the most general harmonic oscillator Hamiltonian is identified (incorrectly) as the energy, both the generalized Berry phase and the dynamical phase are different from those calculated from the energy operator. The Hamiltonian in this case is not the energy, because its time derivative is not the power. Nevertheless, in the adiabatic limit the *total* phase is the same as when the correct energy operator is used, which is an example of a general theorem [10]. However, since both the dynamical phase and the Berry phase are measurable, it makes a difference whether eigenstates of the energy operator or the Hamiltonian are used. The energy operator gives both the

correct dynamical phase and the generalized Berry phase. The energy spectrum can be determined by spectroscopy and hence the dynamical phase can be calculated. The relative total phase can be measured by an interference experiment, so the generalized Berry phase can be obtained by subtraction of the dynamical phase.

In section 2 the most general Hamiltonian for the harmonic oscillator is used to obtain the classical equation of motion and the energy of the oscillator. In section 3 the generalized Berry phase is shown to have both a state-dependent part and an arbitrary state-independent part In section 4 the Hamiltonian is assumed to be the energy, which results in a generalized Berry phase with a different state-dependent part. The energies and total phases obtained in sections 3 and 4 are compared in section 5, along with a discussion of the uncertainty principle The conclusion is given in section 6.

2. Classical most general time-dependent harmonic oscillator

2.1. Hamiltonian and equations of motion

The most general time-dependent Hamiltonian for the harmonic oscillator is both linear and quadratic in the generalized coordinate q and conjugate canonical momentum p:

$$H = \tfrac{1}{2}(ap^2 + 2bpq + cq^2) + fq + gp + h \tag{2.1}$$

where the coefficients a, b, c, f, g and h are all time dependent (but not necessarily adiabatically varying in time) The classical equation of motion is obtained from Hamilton's equations:

$$\dot{q} = \partial H/\partial p = ap + bq + g \tag{2.2}$$

and

$$\dot{p} = -\partial H/\partial q = -b\dot{q} - cq - f. \tag{2.3}$$

In order to obtain the classical equation of motion, it is necessary to know the, possibly time-dependent, mass $m(t)$ of the oscillator and the positive or negative time-dependent friction 'constant' $\gamma(t)$. Since the Hamiltonian in equation (2.1) does not give this physical information, which must be known *a priori*, the Hamiltonian alone does not completely determine the physical situation. The function $a(t)$ is related to the mass $m(t)$ and friction 'constant' $\gamma(t)$ by

$$a^{-1} = m(t)\Gamma(t) \tag{2.4}$$

where the function $\Gamma(t)$ is defined as

$$\Gamma(t) = \exp\left(\int_0^t \mathrm{d}t'\, \gamma(t')/m(t')\right). \tag{2.5}$$

The function $\gamma(t)$ describes friction when it is positive and antifriction when it is negative, and henceforth will be referred to as the 'friction constant'.

When equation (2.4) is used in equations (2.2) and (2.3) and the canonical momentum p is eliminated, the classical equation of motion is

$$\mathrm{d}(m\dot{q})/\mathrm{d}t = -\gamma\dot{q} - kq + F \tag{2.6}$$

where the time-dependent spring 'constant' is

$$k(t) = \Gamma^{-1}[c - b^2/a - \mathrm{d}(b/a)/\mathrm{d}t] \tag{2.7}$$

and the time-dependent force $F(t)$ is

$$F(t) = \Gamma^{-1}[\mathrm{d}(g/a)/\mathrm{d}t + gb/a - f]. \tag{2.8}$$

The physical system described by the Hamiltonian in equation (2.1) is thus in general a time-dependent mass $m(t)$ on a spring with a time-dependent spring 'constant' $k(t)$ acted upon by a velocity-dependent force with a time-dependent friction 'constant' $\gamma(t)$ and a time-dependent force $F(t)$. Previously, cases where (i) the mass is constant and $\gamma > 0$ [21], and (ii) the mass is time dependent and $\gamma = 0$ [25, 30] have been considered. The general case has both a time-dependent mass and a time-dependent friction 'constant'

2.2. Energy

The energy $\mathscr{E}$ for the harmonic oscillator described by equation (2.6) is

$$\mathscr{E} = \tfrac{1}{2}m\dot{q}^2 + \tfrac{1}{2}kq^2 - \bar{F}q \tag{2.9}$$

where $\bar{F}$ is the average of the time-dependent force $F(t)$ acting on the particle,

$$\bar{F} = T^{-1}\int_0^T \mathrm{d}t\, F(t) \tag{2.10}$$

as $T \to \infty$. The time rate of change of this energy is the power P transferred between the system and the environment:

$$\mathrm{d}\mathscr{E}/\mathrm{d}t = P = (\tfrac{1}{2}\dot{m}q^2 + \tfrac{1}{2}\dot{k}q^2) + (-m\dot{q})\dot{q} + (-\gamma\dot{q})q + \Delta F q \tag{2.11}$$

where the time-dependent non-conservative force is $\Delta F = F(t) - \bar{F}$. On the right-hand side of equation (2.11), the first term is the power transferred due to the change in time of the mass at constant velocity, the second term is due to the change in time of the spring 'constant' at constant displacement, the third term is due to the 'fictitious force' $-m\dot{q}$, which is taken from the left-hand side of equation (2.6), the fourth term is the power loss (or gain) due to the friction (or antifriction) force, and the last term is the power transferred by the time-dependent non-conservative force ΔF. Therefore, each term in equation (2.11) has a physical interpretation.

When equation (2.2) for the velocity is substituted into equation (2.9) for the energy, and equation (2.4) is used, we obtain

$$\mathscr{E} = (2m\Gamma^2)^{-1}(p + bq/a + g/a)^2 + \tfrac{1}{2}k(q - q_0)^2 - \tfrac{1}{2}kq_0^2 \tag{2.12}$$

where $q_0(t) = \bar{F}/k$ is a parameter obtained by completing the square in equation (2.9). The energy in equation (2.12) is now expressed in terms of the canonical momentum and is in a form suitable for canonical quantization.

3. Generalized Berry phase

3.1 Energy eigenvalue problem

The system discussed in section 2 can be canonically quantized by replacing the canonical momentum p by the canonical momentum operator $\hat{p} = -\mathrm{i}\partial/\partial q$, where natural units such that $\hbar = 1$ are used.

For the energy in equation (2.12) the energy operator eigenvalue problem is

$$\mathscr{E}\psi_n = \varepsilon_n\psi_n \tag{3.1}$$

where ψ_n is the energy eigenstate and ε_n is the energy eigenvalue We can solve equation (3.1) by making a gauge transformation [17] with a gauge function

$$\Lambda(q, t) = (b/2a)q^2 + (g/a)q + r(t) \tag{3.2}$$

where $r(t)$ is an arbitrary function of time. Then the new energy operator $\mathscr{E}'$ is

$$\mathscr{E}' = \exp(i\Lambda)\mathscr{E}\exp(-i\Lambda) = (2m\Gamma^2)^{-1}p^2 + \tfrac{1}{2}k(q-q_0)^2 - \tfrac{1}{2}kq_0^2. \tag{3.3}$$

The new eigenstate ψ_n' corresponding to the new energy operator in equation (3.3) is

$$\psi_n' = \exp(i\Lambda)\psi_n = \phi_n(q-q_0). \tag{3.4}$$

The function $\phi_n(q)$ is the real energy eigenstate of the simple harmonic oscillator, whose energy operator is equation (3.3) with $q_0 = 0$ The eigenenergy ε_n in equation (3.1) is unchanged under a gauge transformation, and is

$$\varepsilon_n = (n+\tfrac{1}{2})\omega - \tfrac{1}{2}kq_0^2 \tag{3.5}$$

where the angular frequency ω is

$$\omega = (k/m\Gamma^2)^{1/2} \tag{3.6}$$

and $n = 0, 1, 2, 3, \ldots$.

3.2. Generalized geometrical phase

The geometrical phase in quantum mechanics discovered by Berry [1] has been generalized to be invariant under unitary transformations and is not limited to adiabatic and cyclic processes [10]. The time derivative of this generalized Berry phase (or Yang [17] phase) is [10]

$$\gamma_n(t) = \langle\psi_n|(i\partial/\partial t - H + \mathscr{E})\psi_n\rangle \tag{3.7}$$

Under a gauge transformation, equation (3.7) is invariant:

$$\dot{\gamma}_n(t) = \langle\psi_n'|(i\partial/\partial t - H' + \mathscr{E}')\psi_n'\rangle \tag{3.8}$$

where the gauge-transformed energy operator and energy eigenstate are defined in equations (3.3) and (3.4), respectively. The gauge-transformed Hamiltonian H' is

$$H' = \exp(i\Lambda)H\exp(-i\Lambda) - \partial\Lambda/\partial t \tag{3.9}$$

which is required for the time-dependent Schrödinger equation to be form invariant [17, 18].

When equations (2.1) and (3.2) are used in equation (3.9), we obtain

$$H' = (2m\Gamma)^{-1}p^2 + \tfrac{1}{2}\Gamma kq^2 - \Gamma Fq + h_0 \tag{3.10}$$

where $h_0(t) = h - g^2/2a - r$ is a function of time only. The time derivative of the Berry phase in equation (3.8) can be written as

$$\dot{\gamma}_n(t) = \langle\psi_n'|(i\partial/\partial t)\psi_n'\rangle - \langle\psi_n'|H'\psi_n'\rangle + \varepsilon_n = (n+\tfrac{1}{2})\omega(1-\Gamma) + h_1(t) \tag{3.11}$$

where $h_1 = h_0 - \Gamma Fq_0 + \tfrac{1}{2}kq_0^2(\Gamma - 1)$ is an arbitrary function of time because $r(t)$ in h_0 is arbitrary The term $\langle\psi_n'|\dot{\psi}_n'\rangle$, where the overdot denotes the partial time derivative,

vanishes because it is both real, from equation (3.4), and imaginary, from the normalization condition $\langle\psi_n'|\psi_n'\rangle = 1$. The generalized Berry phase in equation (3.11) therefore has a state-dependent part, which vanishes if there is no damping ($\Gamma = 1$), and an arbitrary state-independent part h_1. The state-independent part of the generalized Berry phase is chosen such that the total phase has no state-independent part. It has no physical significance because it can be removed by a time-dependent gauge transformation.

3.3. Total phase

In the adiabatic limit the total phase θ of the eigenstate ψ_n is the sum of the dynamical phase and the generalized Berry phase

$$\theta(t) = -\int_0^t \mathrm{d}t'\, \varepsilon_n(t') + \gamma_n(t). \tag{3.12}$$

When equations (3.5) and (3.11) are substituted into equation (3.12), we obtain

$$\theta(t) = -(n+\tfrac{1}{2})\int_0^t \mathrm{d}t'\, \omega(t')\Gamma(t'). \tag{3.13}$$

The arbitrary function $r(t)$ is chosen such that $h_1 + \frac{1}{2}kq_0^2 = 0$, so that the total phase does not have any state-independent contribution. If there is no damping ($\Gamma = 1$), equation (3.13) gives the dynamical phase.

4. Hamiltonian as the energy

4.1. Hamiltonian

If we assume that the Hamiltonian in equation (2.1) is the energy of the particle, we obtain a non-zero generalized Berry phase different from section 3. The Hamiltonian in equation (2.1) can be rewritten as

$$H = \tilde{\mathscr{E}} = (2m\Gamma)^{-1}(p + bq/a + g/a)^2 + \tfrac{1}{2}\Gamma\tilde{k}(q - \tilde{q}_0)^2 + \tilde{h}_0 \tag{4.1}$$

where $\tilde{h}_0 = h - g^2/2a - \frac{1}{2}\Gamma\tilde{k}\tilde{q}_0^2$ is a function of time only and $\tilde{q}_0 = \tilde{F}/\tilde{k}$. The time-dependent 'spring constant' $\tilde{k}$ is

$$\tilde{k} = \Gamma^{-1}(c - b^2/a) \tag{4.2}$$

and

$$\tilde{F} = \Gamma^{-1}(gb/a - f). \tag{4.3}$$

The eigenvalue problem for $\tilde{\mathscr{E}}$ is

$$\tilde{\mathscr{E}}\tilde{\psi}_n = \tilde{\varepsilon}_n\tilde{\psi}_n \tag{4.4}$$

where $\tilde{\psi}_n$ is the eigenstate and $\tilde{\varepsilon}_n$ is the eigenvalue. Equation (4.4) can be immediately solved if we make a gauge transformation with the gauge function

$$\tilde{\Lambda}(q, t) = (b/2a)q^2 + (g/a)q + \tilde{r}(t) \tag{4.5}$$

where $\tilde{r}$ is an arbitrary function of time. Then the new operator $\tilde{\mathscr{E}}'$ is

$$\tilde{\mathscr{E}}' = \exp(\mathrm{i}\tilde{\Lambda})\tilde{\mathscr{E}}\exp(-\mathrm{i}\tilde{\Lambda}) = (2m\Gamma)^{-1}p^2 + \tfrac{1}{2}\Gamma\tilde{k}(q - \tilde{q}_0)^2 + \tilde{h}_0. \tag{4.6}$$

The new eigenstate $\tilde{\psi}'_n$ of the new operator in equation (4.6) is

$$\tilde{\psi}'_n = \exp(i\tilde{\Lambda})\tilde{\psi}_n = \tilde{\phi}_n(q - \tilde{q}_0). \tag{4.7}$$

The function $\tilde{\phi}_n(q)$ is the real eigenstate to the simple harmonic oscillator with $\tilde{q}_0 = 0$ in the Hamiltonian in equation (4.6). The eigenvalue in equation (4.4) is

$$\tilde{\varepsilon}_n = (n + \tfrac{1}{2})\tilde{\omega} + \tilde{h}_0(t) \tag{4.8}$$

where the angular frequency $\tilde{\omega}$ is

$$\tilde{\omega} = (\tilde{k}/m)^{1/2} \tag{4.9}$$

and $n = 0, 1, 2, 3, \ldots$

4.2. Generalized geometrical phase

The time derivative of the generalized Berry phase is [10]

$$\dot{\tilde{\gamma}}_n(t) = \langle \tilde{\psi}_n | (i\partial/\partial t - H + \tilde{\mathscr{E}})\tilde{\psi}_n \rangle. \tag{4.10}$$

Since the energy operator is equal to the Hamiltonian in this case, equation (4.10) reduces to

$$\dot{\tilde{\gamma}}_n(t) = \langle \tilde{\psi}'_n | (\partial\tilde{\Lambda}/\partial t)\tilde{\psi}'_n \rangle + \langle \tilde{\psi}'_n | (i\partial/\partial t)\tilde{\psi}'_n \rangle \tag{4.11}$$

with the use of equation (4.7). The last term on the right-hand side of equation (4.11) is zero, because it is both real, from equation (4.7), and imaginary from the normalization condition. If equation (4.5) for the gauge function is substituted into equation (4.11), we obtain

$$\dot{\tilde{\gamma}}_n(t) = \tfrac{1}{2}[\mathrm{d}(b/a)/\mathrm{d}t](a/\tilde{\omega})(n + \tfrac{1}{2}) + \dot{\tilde{r}} + \tilde{q}_0\, \mathrm{d}(g/a)/\mathrm{d}t \tag{4.12}$$

which has a state-dependent part and an arbitrary state-independent part. The state-dependent part of the generalized Berry phase in equation (4.12) is different from the state-dependent part of the generalized Berry phase in equation (3.11).

4.3. Total phase

In the adiabatic limit the total phase $\tilde{\theta}$ of the eigenstate $\tilde{\psi}_n$ is the sum of the dynamical phase and the generalized Berry phase:

$$\tilde{\theta}(t) = -\int_0^t \mathrm{d}t'\, \tilde{\varepsilon}_n(t') + \tilde{\gamma}_n(t). \tag{4.13}$$

When equations (4.8) and (4.12) are substituted into equation (4.13), we obtain

$$\tilde{\theta}(t) = -(n + \tfrac{1}{2}) \int_0^t \mathrm{d}t' [\tilde{\omega}(t') - \tfrac{1}{2}(a/\tilde{\omega})\, \mathrm{d}(b/a)/\mathrm{d}t']. \tag{4.14}$$

The arbitrary function $\tilde{r}$ is chosen to be $\dot{\tilde{r}} = \tilde{h}_0 - \tilde{q}_0\, \mathrm{d}(g/a)/\mathrm{d}t$, so that equation (4.14) has no state-independent contribution.

5. Discussion

In this section a comparison of the energies and total phases in sections 3 and 4 is made. The uncertainty principle is also discussed.

5.1. Energy

The time derivative of the energy $\mathscr{E}$ in equation (2.9) is the power in equation (2.11) transferred between the system and its environment. Each term in equation (2.11) has a physical significance

On the other hand, if the 'energy' is chosen to be the Hamiltonian H in equation (2.1), then the time derivative of $\tilde{\mathscr{E}}$ in equation (4.1) is

$$\mathrm{d}\tilde{\mathscr{E}}/\mathrm{d}t = \mathrm{d}H/\mathrm{d}t = \partial H/\partial t = \tfrac{1}{2}(\dot{a}p^2 + 2\dot{b}pq + \dot{c}q^2) + \dot{f}q + \dot{g}p + \dot{h} \tag{5.1}$$

from Hamilton's equations (2.2) and (2.3). If the canonical momentum p is eliminated from equation (5.1) in terms of the velocity $\dot{q}$ using equation (2.2), we obtain

$$\mathrm{d}\tilde{\mathscr{E}}/\mathrm{d}t = -\tfrac{1}{2}[\mathrm{d}(m\Gamma)/\mathrm{d}t]\dot{q}^2 + \tfrac{1}{2}[\mathrm{d}(\Gamma\tilde{k})/\mathrm{d}t]q^2 - [\mathrm{d}(\Gamma\tilde{F})/\mathrm{d}t]q + [\mathrm{d}(b/a)/\mathrm{d}t]qq$$
$$-[\mathrm{d}(g/a)/\mathrm{d}t]\dot{q} + \mathrm{d}(h - g^2/2a)/\mathrm{d}t \tag{5.2}$$

The six terms on the right-hand side of equation (5.2) cannot be interpreted physically as power, in contrast to the five terms on the right-hand side of equation (2.11). We can thus conclude that $\tilde{\mathscr{E}}$ in equation (4.1) is not the energy of the system, but $\mathscr{E}$ in equation (2.9) is.

The difference between the energy $\mathscr{E}$ in equation (2.9) and $\Gamma^{-1}\tilde{\mathscr{E}}$ in equation (4.1) is

$$\mathscr{E} - \Gamma^{-1}\tilde{\mathscr{E}} = \tfrac{1}{2}(k - \tilde{k})q^2 - (\tilde{F} - \tilde{F})q - \Gamma^{-1}(h - g^2/2a) \tag{5.3}$$

In general, the right-hand side of equation (5.3) does not vanish and $\tilde{\mathscr{E}}$ is not the energy.

5.2. Total phase

The total phases in the adiabatic limit for the two cases considered are given in equations (3.13) and (4.14). In the adiabatic limit we now show that these two phases are equal.

The phase $\theta(t)$ in equation (3.13) can be written as

$$\theta(t) = -(n + \tfrac{1}{2}) \int_0^t \mathrm{d}t'[\tilde{\omega}^2 - a\ \mathrm{d}(b/a)/\mathrm{d}t']^{1/2} \tag{5.4}$$

from equations (3.6), (2.4), (2.7), (4.2) and (4.9). In the adiabatic limit, the derivative in equation (5.4) is small, and we can expand the square root, which gives

$$\theta(t) = -(n + \tfrac{1}{2}) \int_0^t \mathrm{d}t'[\tilde{\omega}(t') - \tfrac{1}{2}(a/\tilde{\omega})\ \mathrm{d}(b/a)/\mathrm{d}t' + \ldots]. \tag{5.5}$$

If the expansion is terminated after the second term, equation (5.5) agrees with (4.14), so the two phases are equal: $\theta(t) = \tilde{\theta}(t)$. This behaviour is an example of a general theorem [10].

Even though the total phases in sections 3 and 4 are the same in the adiabatic limit, the dynamical and Berry phases are different. The energies of a system can be determined from spectroscopic measurements, and the dynamical phase can be calculated. The dynamical phase in section 3 is correct, because the correct energy operator is used. An interference experiment would measure the total relative phase. The Berry phase can be obtained by subtracting the dynamical phase from the total phase. Because the correct energy is used, the Berry phase obtained in section 3 is correct.

5.3. Uncertainty principle

For a system which is canonically quantized, the uncertainty principle is

$$\Delta p \Delta q \geq \hbar/2 \tag{5.6}$$

in terms of the uncertainties in the canonical momentum $\Delta p = \langle (p - \langle p \rangle)^2 \rangle^{1/2}$ and the coordinate $\Delta q = \langle (q - \langle q \rangle)^2 \rangle^{1/2}$. For the damped harmonic oscillator, the uncertainty principle in terms of the kinetic momentum $mv = mq$ from equation (2.2) is

$$\Delta mv \Delta q \geq (\hbar/2) \exp\left(- \int_0^t dt' \, \gamma(t')/m(t') \right) \tag{5.7}$$

where equations (2.4) and (2.5) have been used.

The Kanai Hamiltonian [21], for which $\gamma/m > 0$ is a constant in equation (5.7), has been criticized [23] for allegedly violating the uncertainty principle. However, the uncertainty principle in terms of the canonical momentum p in equation (5.6) is always satisfied. Nevertheless, some workers [23] believe on physical grounds that the uncertainty principle should also be satisfied in terms of the kinetic momentum. When $\gamma/m > 0$, the right-hand side of equation (5.7) goes to zero as the time goes to infinity. This behaviour is due to the model considered, in which the quantum harmonic oscillator is coupled to a reservoir which always removes energy, including the zero point energy as equations (3.5) and (3.6) show. The quantum harmonic oscillator behaves more classically as the time increases.

In the model considered here the 'constant' $\gamma(t)$ depends, in general, on time. It can be positive, corresponding to friction, or negative, corresponding to antifriction. If the integral in equation (5.7) satisfies

$$\int_0^t dt' \, \gamma(t')/m(t') \leq 0 \tag{5.8}$$

for all times t, the right-hand side of equation (5.7) is always greater than or equal to $\hbar/2$, and hence the uncertainty principle in terms of the kinetic momentum $\Delta mv \Delta q \geq \hbar/2$ is always satisfied. The harmonic oscillator with the condition in equation (5.8) gives a phenomenological description of a quantum oscillator which is coupled to a quantum system of many degrees of freedom [24] which can supply energy to the particle as well as remove energy from it. When the reservoir is removing energy from the particle $\gamma > 0$ and when the reservoir is supplying energy to the particle $\gamma < 0$. The model given here does not determine $\gamma(t)$ but merely treats it as a phenomenological function. This approach is a natural generalization of Kanai's Hamiltonian, which removes a major objection [23].

6. Conclusion

The most general time-dependent Hamiltonian for the harmonic oscillator is both linear and quadratic in the coordinate and canonical momentum. It describes a harmonic oscillator with a time-dependent mass, a time-dependent friction (or antifriction) 'constant', and a time-dependent spring 'constant', which is acted upon by a time-dependent force. The time-dependent mass and the time-dependent friction constant must be given to determine the physical system, since the Hamiltonian alone does not do it. The energy, defined such that its time derivative is equal to the power

transferred between the system and the environment, is not in general equal to the Hamiltonian.

When the system is canonically quantized, its generalized Berry phase, calculated for eigenstates of the energy operator, consists of a state-dependent part and an arbitrary state-independent part. The state-independent part is chosen to cancel the state-independent part of the dynamical phase in the adiabatic limit. The state-independent part of the Berry phase is equivalent to a time-dependent gauge function, so it has no observable consequences, contrary to a recent claim [14]. The state-dependent part of the Berry phase contributes only if the system has frictional forces.

If the Hamiltonian is identified (incorrectly) as the energy, the generalized Berry phase has a different state-dependent part and an arbitrary state-independent part. The state-independent part of the Berry phase can also be chosen to cancel the state-independent part of the dynamical phase. In the adiabatic limit the total phases obtained from the two choices of energy are the same [10]. Nevertheless, the dynamical phases and the Berry phases are different. Since the energy can be measured, the dynamical phase can be calculated. It is therefore essential to use the correct energy operator for the problem.

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